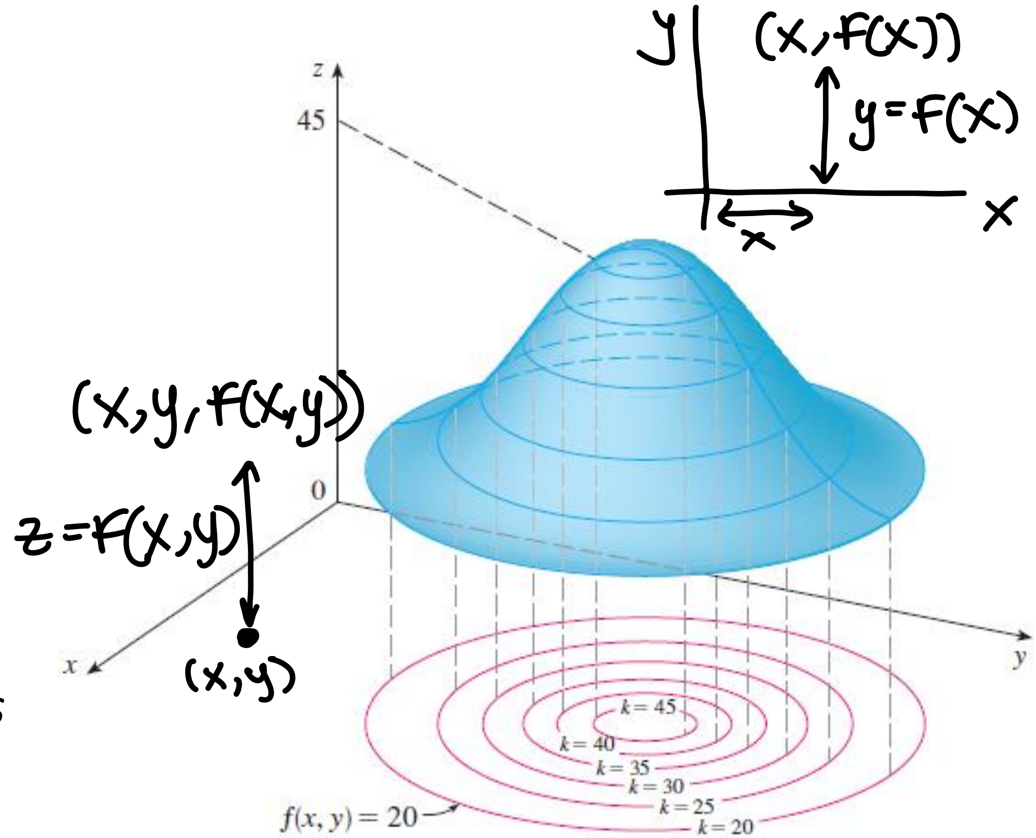
## 14.1 & 14.3 Intro to Surfaces and Partial Derivatives

*Def'n*: A function, *f*, of two variables is assigns a number for each input (x,y)

$$z = f(x, y).$$

*(x,y)* = location on the xy-plane.
*z = f(x,y)* = height above that pt.




Our main visualization tool is *level curves* (traces for *z = k,* a constant)

Domain of a function = restrictions, where it works

$\sqrt{\text{Blah}} \Rightarrow \text{Blah} \geq 0$

$y = \sqrt{5-x} \Rightarrow 5-x \geq 0$

$x \leq 5$

$\frac{1}{\text{stuff}} \Rightarrow \text{stuff} \neq 0$

$\ln(\text{things}) \Rightarrow \text{things} > 0$

Find and sketch domain of...

$z = \ln(4 - x^2 - 4y^2)$

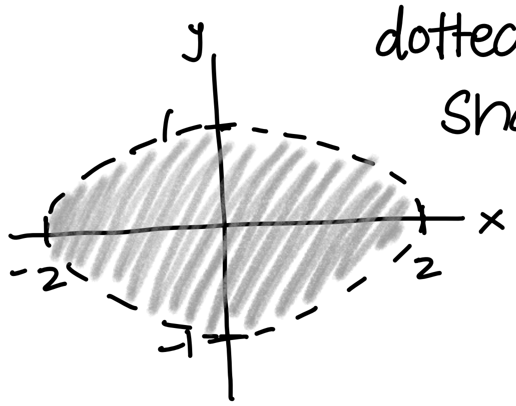
$4 - x^2 - 4y^2 > 0$ ⇐ Region, not an interval

$4 - x^2 - 4y^2 = 0$ ⇐ ellipse!

$x^2 + 4y^2 = 4$ draw an ellipse...

dotted line = doesn't include those points

shade inside (not outside)




$z = \sqrt{9 - x^2} - \sqrt{25 - y^2}$

$$\left.\begin{aligned} \sqrt{9 - x^2} &\geq 0 \\ \sqrt{25 - y^2} &\geq 0 \end{aligned}\right\}$$ both happening

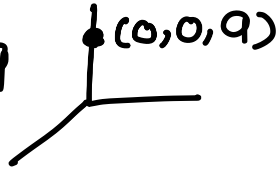
circular paraboloid

Graph several level curves for

$$z = f(x, y) = 9 - x^2 - y^2$$

and make a contour map.

$f(0,0) = 9$

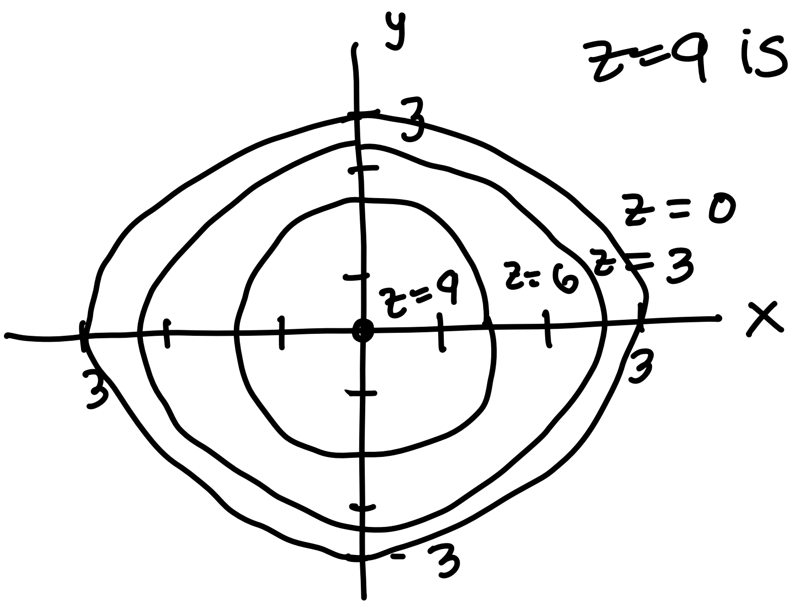



$z = 0 \rightarrow \quad 0 = 9 - x^2 - y^2 \quad x^2 + y^2 = 9$

$z = 3 \rightarrow \quad 3 = 9 - x^2 - y^2 \quad x^2 + y^2 = 6$

$z = 6 \rightarrow \quad 6 = 9 - x^2 - y^2 \quad x^2 + y^2 = 3$

$z = 9 \rightarrow \quad 9 = 9 - x^2 - y^2 \quad x^2 + y^2 = 0$




z=9 is a high point/a maximum

(topographic map)

***Aside: Tips for identifying graphs in HW:***

1. Plot a few points…
   - Like f(0,0), f(0,1), f(1,0)
2. Draw a few level curves…
   - Like z = 0, z = 1, z = 2
3. Special features?
   - Places it is undefined
   - Behavior as x and y get bigger
   - Is it a wave if x (or y) is fixed?

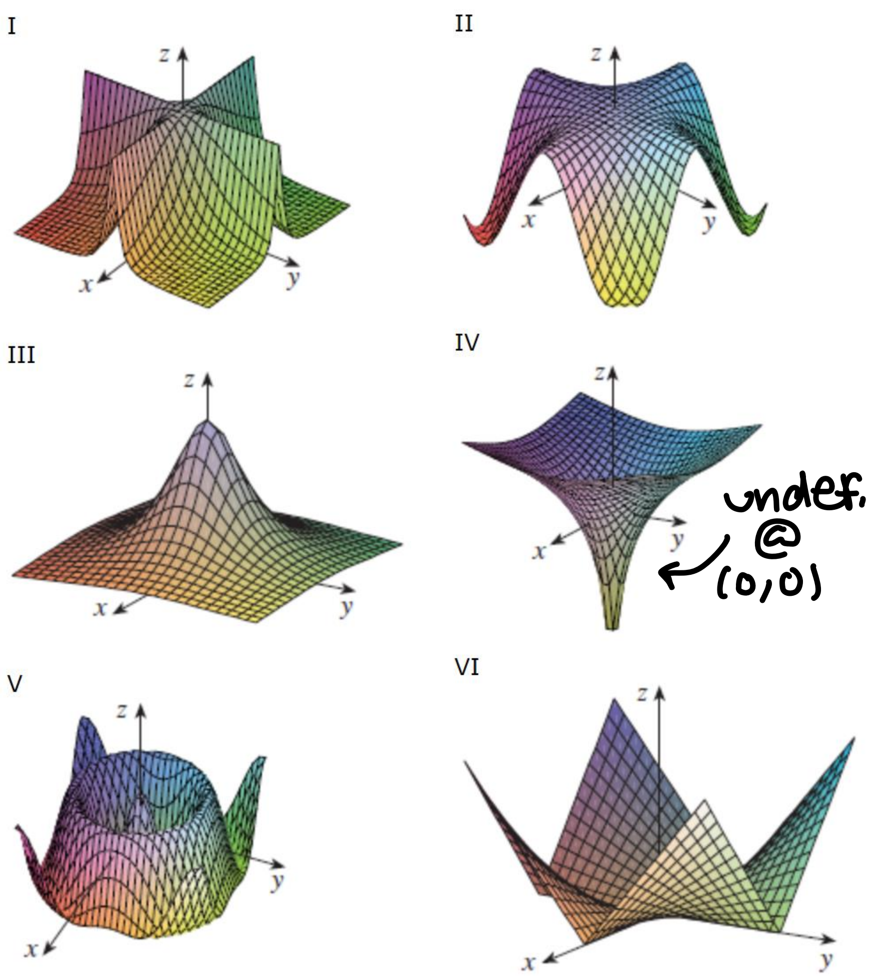
*Visuals:* https://www.math3d.org/vWjudPnx

# HW 14.1 / 9:

*Match the function to the picture*

$1 = \frac{1}{1+x^2+y^2} \Rightarrow x^2+y^2 = 0$

circle w/ point




undef. @ (0,0)

(a) $f(x, y) = \dfrac{1}{1 + x^2 + y^2}$

? ⌄

(b) $f(x, y) = \dfrac{1}{1 + x^2y^2}$

? ⌄

(c) $f(x, y) = \ln(x^2 + y^2)$ ← undef. @ (0,0)

? ⌄

← wave

(d) $f(x, y) = \cos\left(\sqrt{x^2 + y^2}\right)$

? ⌄

$0 = \cos\sqrt{x^2+y^2}$

$\frac{\pi}{2} = \sqrt{x^2+y^2}$

$\frac{\pi^2}{4} = x^2+y^2$

circles

(e) $f(x, y) = |xy|$

? ⌄

← wave

(f) $f(x, y) = \cos(xy)$

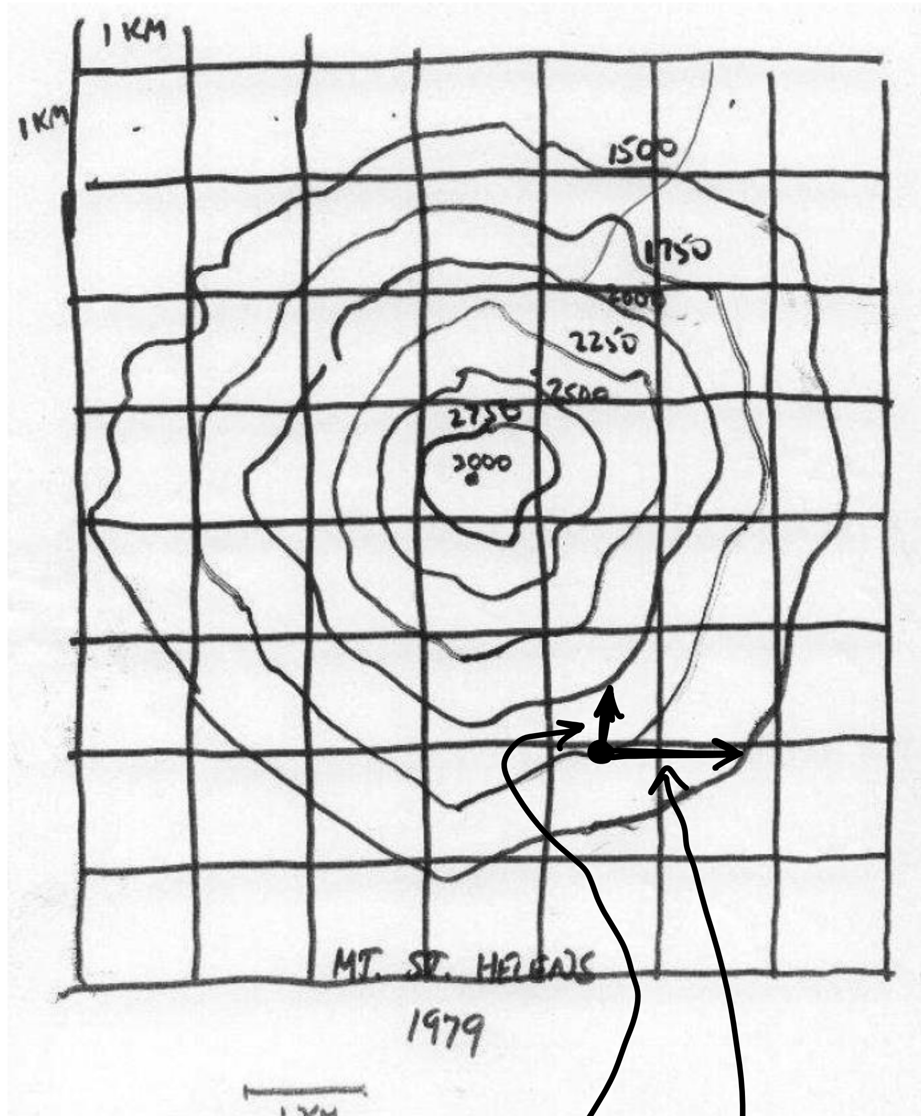
? ⌄

### *Contour Maps*

If we are looking at *z = c traces*, we call them **level curves**. A collection of level curves at equally spaced z values is called a **contour map**.

What does it mean if...

- contours are closer together? steeper
- Contours cross? pringles chip




both slopes = 0, @ top

slope = $\frac{d}{dy}$ slope = $\frac{d}{dx}$

**14.3 Partial Derivatives**

Goal: Get slope in two different directions on a surface.

We define:

$$\frac{\partial z}{\partial x} = f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,\overset{\text{fixed}}{y}) - f(x,y)}{h}$$

$$\frac{\partial z}{\partial y} = f_y(x,y) = \lim_{h\to 0} \frac{f(\overset{\text{fixed}}{x},y+h) - f(x,y)}{h}$$

$\partial$ = partial derivative, not all variables involved

*Quick Example*

$$f(x,y) = x^2y - 9y - xy^2 + y^3$$

$F(x,2) = x^2 \cdot 2 - 9\cdot 2 - x(2)^2 + (2)^3$

$2x\cdot 2 \quad -0 \quad -1\cdot 4 \quad +0$

$f(2,y) = (2)^2 y - 9y - 2y^2 - y^3$

$F_x = 2xy - 0 - 1y^2 + 0 = \boxed{2xy - y^2} \Rightarrow$ slope in x direction

$F_y = \boxed{x^2 - 9 - 2xy + 3y^2} \Rightarrow$ slope in y direction

Visuals: https://www.math3d.org/gVsjaecy

*Example:*

two x's ⇒ product rule

$$f(x,y) = x^3y + \underbrace{x^5e^{xy^2}} + \ln(y)$$

terms ⇒ 3

*Example:*

$$g(x,y) = \cos(x^3 + y^4)$$

$$F_x = 3x^2y + x^5y^2e^{xy^2} + 5x^4e^{xy^2} + 0$$

$$F_y = x^3 + \qquad + \frac{1}{y}$$

**Notes on Variables and Derivatives**

In calculus a variable can be treated as:

1. A constant
2. An independent variable (input)
3. A dependent variable (output),

*Old Examples and Review*:

a) **One variable function of *x***:

Find $\frac{dy}{dx}$ for $y - x^2 = 0.$

$$\frac{dy}{dx} = 2X$$

b) **Related rates**: At time *t* assume a particle is moving along the path $y = x^2$.

Find $\frac{dy}{dt}$. $y(t) = (x(t))^2$

$$\frac{dy}{dt} = 2x\frac{dx}{dt}$$

c) **Implicit**: $x^2 + xy^2 = 1$

Find $\frac{dy}{dx}$.

$$\frac{d}{dx}\left(x^2 + x(y(x))^2 = 1\right)$$

$$2x + x2y\frac{dy}{dx} + y^2 = 0$$

## *****NEW** Examples***

d) **Multivariable**: $z = x^2 + y^3 - 5xy^4$

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial x} = 2x - 5y^4$$

$$\frac{\partial z}{\partial y} = 3y^2 - 20xy^3$$

$$(z(x))^2 = 2(z(x))\frac{dz}{dx}$$

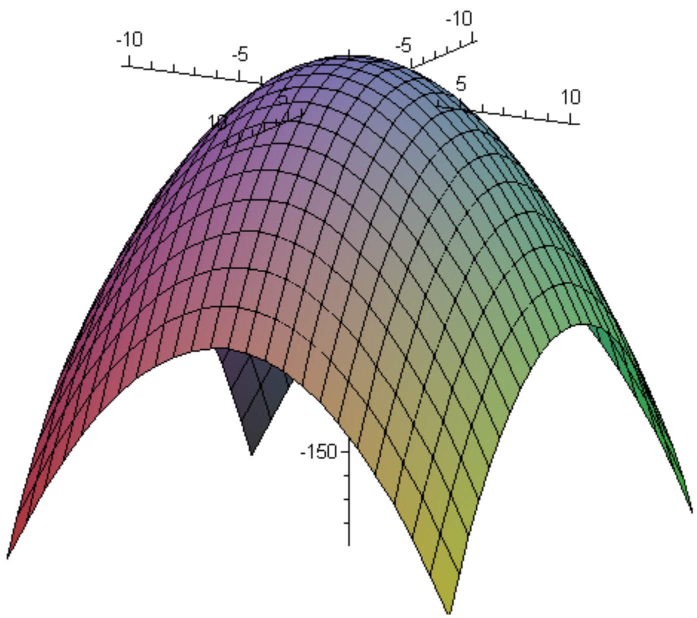
e) **Multi. Implicit**: $x^2 + xy^2 - z^2 = 1$

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial x} \Rightarrow 2x + y^2 - 2z\frac{dz}{dx} = 0 \Rightarrow \boxed{\frac{dz}{dx} = \frac{-2x - y^2}{-2z}}$$

*More on Graphical Interpretation*:
Pretend you are skiing on the surface

$$z = f(x, y) = 15 - x^2 - y^2.$$




*Exercises*:

1. Find $f_x(x, y)$ and $f_y(x, y)$

2. Assume you are standing on the point on the surface corresponding to (x,y) = (4,7).

   Compute:

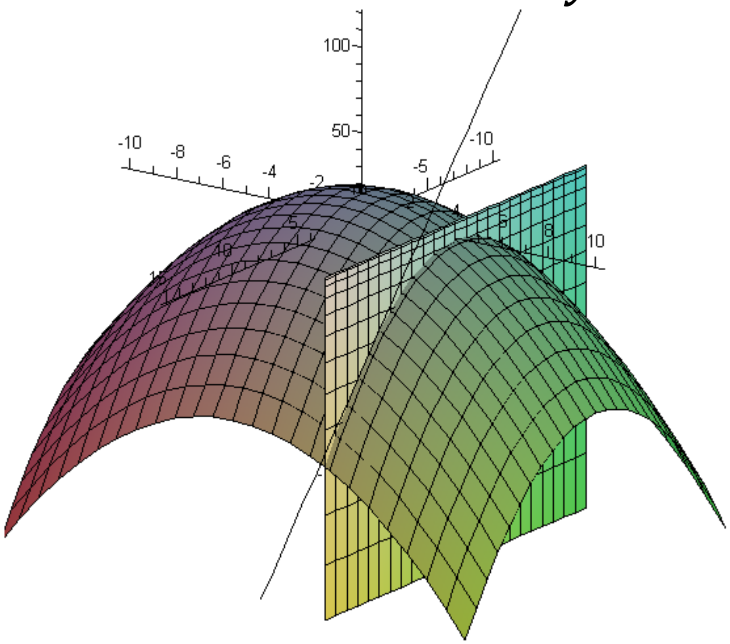
   i) $f(4,7) =$

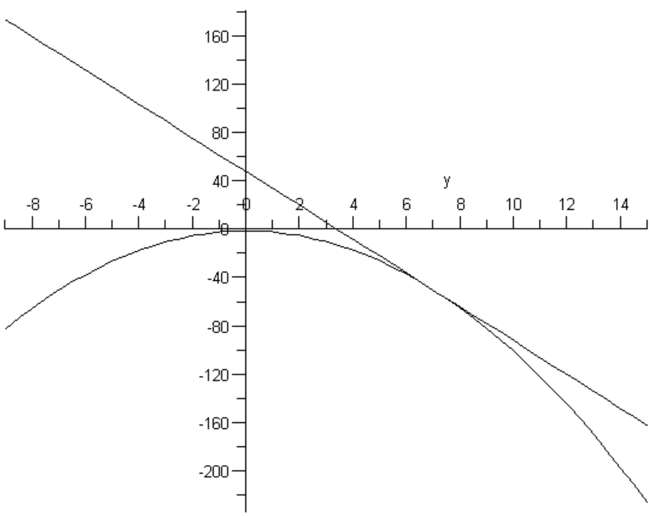
   ii) $f_x(4,7) =$

   iii) $f_y(4,7) =$

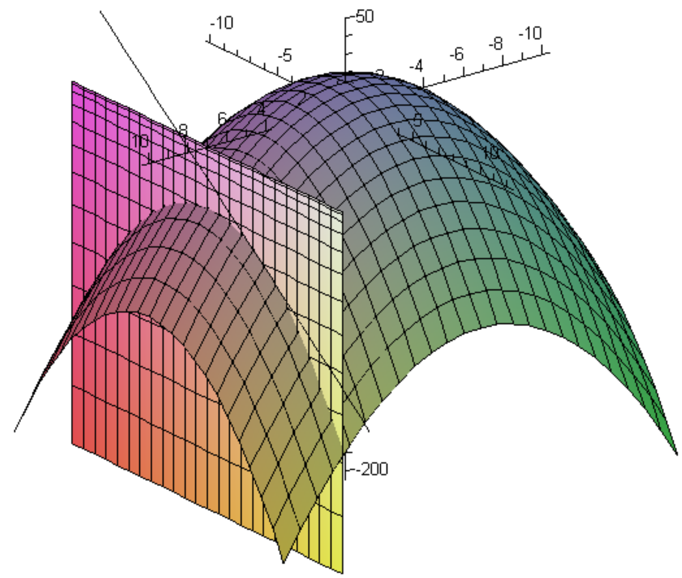
   What do these three numbers represent?

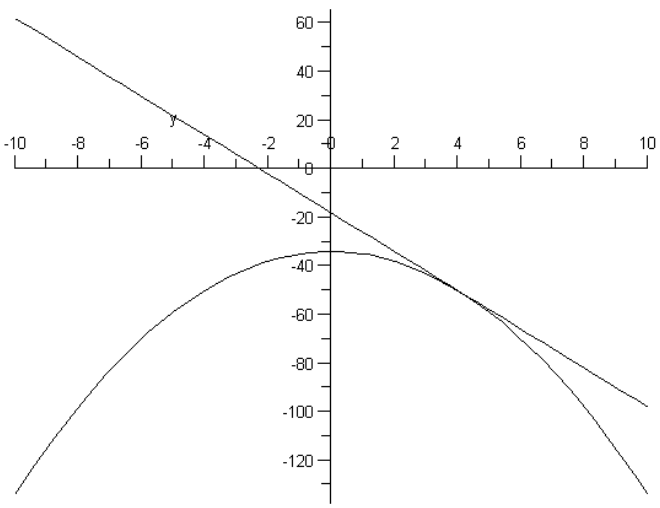
The plane y = 4 intersecting the surface $z = 15 - x^2 - y^2$.






The plane x = 7 intersecting the surface $z = 15 - x^2 - y^2$.





## Second Partial Derivatives

Concavity in *x*-direction:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = f_{xx}(x, y)$$

Concavity in *y*-direction:

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = f_{yy}(x, y)$$

Mixed Partials:

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = f_{xy}(x, y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f_{yx}(x, y)$$

*Example*: Find all second partials for
$$z = f(x, y) = x^4 + 3x^2y^3 + y^5$$